

# Cosmological Bianchi Class A models in Sáez-Ballester theory

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We use the Sáez-Ballester (SB) theory on anisotropic Bianchi Class A cosmological model, with barotropic fluid and cosmological constant, using the Hamilton or Hamilton-Jacobi approach. Contrary to claims in the specialized literature, it is shown that the Sez-Ballester theory cannot provide a realistic solution to the dark matter problem of Cosmology for the dust epoch, without a fine tuning because the contribution of the scalar field in this theory is equivalent to a stiff fluid (as can be seen from the energy-momentum tensor for the scalar field), that evolves in a different way as the dust component. To have similar contributions of the scalar component and the dust component implies that their past values were fine tuned. So, we reinterpreting this null result as an indication that dark matter plays a central role in the formation of structures and galaxy evolution, having measureable effects in the cosmic microwave bound radiation, and than this formalism yield to this epoch as primigenius results. We do the mention that this formalism was used recently in the so called K-essence theory applied to dark energy problem, in place to the dark matter problem. Also, we include a quantization procedure of the theory which can be simplified by reinterpreting the theory in the Einstein frame, where the scalar field can be interpreted as part of the matter content of the theory, and exact solutions to the Wheeler-DeWitt equation are found, employing the Bianchi Class A cosmological models.

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## I. INTRODUCTION

Several observations suggest that in galaxies and galaxy clusters there is an important quantity of matter that is not interacting electromagnetically, but only through gravitation. This is the well known dark matter problem. Several solutions have been considered for this problem, modifying the gravitational theory or introducing new forms of matter and interactions. To address the dark matter problem Saez and Ballester (SB) [1] formulated a scalar-tensor theory of gravitation in which the metric is coupled with a dimensionless scalar field. In a recent analysis using the standard scalar field cosmological models [2, 3], contrary to claims in the specialized literature, it is shown that the SB theory cannot provide a realistic solution to the dark matter problem of Cosmology for the dust epoch, because the contribution of the scalar field is equivalent to stiff matter. We can reinterpret this result in a sense that the galaxy halo was formed during this primigenius epoch and its evolution until the dust era using the standard scalar field cosmological theory. In this theory the strength of the coupling between gravity and the scalar field is determined by an arbitrary coupling constant  $\omega$ . This constant  $\omega$  can be used to have a lorentzian  $(-1,1,1,1)$  or pseudo-lorentzian  $(-1,-1,1,1)$  signature when we build the Wheeler-DeWitt equation. The values for this constant, in the classical regime, are dictated by the condition to have real functions. Other problem inherent to this theory is that not exist how build the invariants with this field as in the case to scalar curvature. So, was necessary to reinterpret the formalism where this field is considered as matter content in the theory in the Einstein frame.

On the other hand, this approach is classified with another name, by instant, Armendariz-Picon et al, called this formalism as K-essence [4], as a dynamical solution for explaining naturally why the universe has entered an epoch of accelerated expansion at a late stage of its evolution. Instead, K-essence is based on the idea of a dynamical attractor solution which causes it to act as a cosmological constant only at the onset of matter domination. Consequently, K-essence overtakes the matter density and induces cosmic acceleration at about the present epoch. Usually K-essence models are restricted to the Lagrangian density of the form

$$S = \int d^4x \sqrt{-g} f(\phi) (\nabla\phi)^2. \quad (1)$$

One of the motivations to consider this type of Lagrangian originates from string theory [5]. For more details for K-essence applied to dark energy, you can see in [6] and reference therein. Many works in SB formalism in the classical regime have been done, where the Einstein field equation is solved in a direct way, using a particular ansatz for the main scalar factor of the universe

[7, 8, 10, 11], yet a study of the anisotropy behaviour through the form introduced in the line element has been conducted [9, 12–20].

On another front, the quantization program of this theory has not been constructed. The main complication can be traced to the lack of an ADM type formalism. We can transform this theory to conventional one where the dimensionless scalar field is obtained from energy-momentum tensor as an exotic matter contribution, and in this sense we can use this formalism for the quantization program, where the ADM formalism is well known [21].

In this work, we use this formulation to obtain classical and quantum exact solutions to anisotropic Bianchi Class A cosmological models with stiff matter. The first step is to write SB formalism in the usual manner, that is, we calculate the corresponding energy-momentum tensor to the scalar field and give the equivalent Lagrangian density. Next, we proceed to obtain the corresponding canonical Lagrangian  $\mathcal{L}_{can}$  to Bianchi Class A cosmological models through the Legendre transformation, we calculate the classical Hamiltonian  $\mathcal{H}$ , from which we find the Wheeler-DeWitt (WDW) equation of the corresponding cosmological model under study. We employ in this work the Misner parametrization due that a natural way appear the anisotropy parameters to the scale factors.

The simpler generalization to Lagrangian density for the SB theory [1] with the cosmological term, is

$$\mathcal{L}_{geo} = (R - 2\Lambda - F(\phi)\phi_{,\gamma}\phi^{,\gamma}), \quad (2)$$

where  $\phi^{,\gamma} = g^{\gamma\alpha}\phi_{,\alpha}$ ,  $R$  the scalar curvature,  $F(\phi)$  a dimensionless function of the scalar field.. In classical field theory with scalar field, this formalism corresponds to null potential in the field  $\phi$ , but the kinetic term is exotic by the factor  $F(\phi)$ .

From the Lagrangian (2) we can build the complete action

$$I = \int_{\Sigma} \sqrt{-g}(\mathcal{L}_{geo} + \mathcal{L}_{mat})d^4x, \quad (3)$$

where  $\mathcal{L}_{mat}$  is the matter Lagrangian,  $g$  is the determinant of metric tensor. The field equations for this theory are

$$G_{\alpha\beta} + g_{\alpha\beta}\Lambda - F(\phi) \left( \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma} \right) = -8\pi G T_{\alpha\beta}, \quad (4a)$$

$$2F(\phi)\phi^{,\alpha}_{;\alpha} + \frac{dF}{d\phi}\phi_{,\gamma}\phi^{,\gamma} = 0, \quad (4b)$$

where  $G$  is the gravitational constant and as usual the semicolon means a covariant derivative.

The equation (4b) take the following form for all cosmological Bianchi Class A models, assuming that the scalar field is only time dependent ( here  $\prime = \frac{d}{d\tau} = \frac{d}{Ndt}$ )

$$3\Omega'\phi'F + \phi''F + \frac{1}{2}\frac{dF}{d\phi}\phi'^2 = 0$$

which can be put in quadrature form as

$$\frac{1}{2}F\phi'^2 = F_0e^{-6\Omega}, \quad (5)$$

this equation is seen as corresponding to a stiff matter content contribution.

The same set of equations(4a,4b) is obtained if we consider the scalar field  $\phi$  as part of the matter budget, i.e. say  $\mathcal{L}_\phi = -F(\phi)g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$  with your corresponding energy-momentum tensor

$$T_{\alpha\beta} = F(\phi) \left( \phi_{,\alpha}\phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}\phi_{,\gamma}\phi^{,\gamma} \right). \quad (6)$$

which is conserved directly considering a stiff matter era in a barotropic scalar fluid (see appendix section 8). In this new line of reasoning, action (3) can be rewritten as a geometrical part (Hilbert-Einstein with  $\Lambda$ ) and matter content (usual matter plus a term that corresponds to the exotic scalar field component of SB theory).

In this way, we write the action (3) in the usual form

$$I = \int_{\Sigma} \sqrt{-g} (R - 2\Lambda + \mathcal{L}_{\text{mat}} + \mathcal{L}_\phi) d^4x, \quad (7)$$

and consequently, the classical equivalence between the two theories. We can infer that this correspondence also is satisfied in the quantum regime, so we can use this structure for the quantization program, where the ADM formalism is well known for different classes of matter [21]. Using this action we obtain the Hamiltonian for SB. We find that the WDW equation is solved when we choose one ansatz similar to this employed in the Bohmian formalism of quantum mechanics and the gravitational part in the solutions are the same that these found in the literature, years ago [22].

This work is arranged as follow. In section 3 we present the method used, employing the FRW cosmological model with barotropic perfect fluid and cosmological constant. In section 4 we construct the Lagrangian and Hamiltonian densities for the anisotropic Bianchi Class A cosmological model. In section 5 the classical solutions using the Jacobi formalism are found. Here we present partial results in the solutions for some Bianchi's cosmological models. Classical solution to Bianchi I is complete in any gauge, but the Bianchi II and  $VI_{h=-1}$ , the solutions are found in particular gauge. Other Biachi's, only the master equation are presented. In Section 6 the complete cuantiza-tion scheme is presented, obtaining the corresponding Wheeler-DeWitt equation and its solutions

are presented in unified way using the classification scheme of Ellis and MacCallum [32] and Ryan and Shepley, [33].

## II. THE METHOD

Let us start with the line element for a homogeneous and isotropic FRW universe

$$ds^2 = -N^2(t)dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right], \quad (8)$$

where  $a(t)$  is the scale factor,  $N(t)$  is the lapse function, and  $\kappa$  is the curvature constant that can take the values 0, 1 and  $-1$ , for flat, closed and open universe, respectively. The total Lagrangian density then reads

$$\mathcal{L} = \frac{6\dot{a}^2 a}{N} - 6\kappa N a + \frac{F(\phi)a^3}{N} \dot{\phi}^2 + 16\pi G N a^3 \rho - 2N a^3 \Lambda, \quad (9)$$

where  $\rho$  is the energy density of matter, we will assume that it complies with a barotropic equation of state of the form  $p = \gamma\rho$ , where  $\gamma$  is a constant. The matter content is assumed as a perfect fluid  $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + g_{\mu\nu}p$  where  $u_\mu$  is the fluid four-velocity satisfying  $u_\mu u^\mu = -1$ . Taking the covariant derivative we obtain the relation

$$3\dot{\Omega}\rho + 3\dot{\Omega}p + \dot{\rho} = 0,$$

whose solution becomes

$$\rho = \rho_\gamma e^{-3\Omega(1+\gamma)}. \quad (10)$$

where  $\rho_\gamma$  is an integration constant.

From the canonical form of the Lagrangian density (9), and the solution for the barotropic fluid equation of motion, we find the Hamiltonian density for this theory, where the momenta are defined in the usual way  $\Pi_{q^i} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ , where  $q^i = (a, \phi)$  are the field coordinates for this system,

$$\begin{aligned} \Pi_a &= \frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{12a\dot{a}}{N}, & \rightarrow & \dot{a} = \frac{N\Pi_a}{12a}, \\ \Pi_\phi &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{2Fa^3\dot{\phi}}{N}, & \rightarrow & \dot{\phi} = \frac{N\Pi_\phi}{2Fa^3}, \end{aligned} \quad (11)$$

so, the Hamiltonian density become

$$\mathcal{H} = \frac{a^{-3}}{24} \left[ a^2 \Pi_a^2 + \frac{6}{F(\phi)} \Pi_\phi^2 + 144\kappa a^4 + 48a^6 \Lambda - 384\pi G \rho_\gamma a^{3(1-\gamma)} \right]. \quad (12)$$

Using the transformation  $\Pi_q = \frac{dS_q}{dq}$ , the Einstein-Hamilton-Jacobi (EHJ) associated to Eq. (12) is

$$a^2 \left( \frac{dS_a}{da} \right)^2 + \frac{6}{F(\phi)} \left( \frac{dS_\phi}{d\phi} \right)^2 + 48a^6 \Lambda - 384\pi G \rho_\gamma a^{3(1-\gamma)} = 0, . \quad (13)$$

The EHJ equation can be further separated in the equations

$$\frac{6}{F(\phi)} \left( \frac{dS_\phi}{d\phi} \right)^2 = \mu^2, \quad (14)$$

$$a^2 \left( \frac{dS_a}{da} \right)^2 + 48a^6 \Lambda - 384\pi G \rho_\gamma a^{3(1-\gamma)} = -\mu^2, \quad (15)$$

where  $\mu$  is a separation constant. With the help of Eqs. (11), we can obtain the solution up to quadratures of Eqs. (14) and (15),

$$\int \sqrt{F(\phi)} d\phi = \frac{\mu}{2\sqrt{6}} \int a^{-3}(\tau) d\tau, \quad (16a)$$

$$\Delta\tau = \int \frac{a^2 da}{\sqrt{\frac{8}{3}\pi G \rho_\gamma a^{3(1-\gamma)} - \frac{\Lambda}{3}a^6 - \nu^2}}, \quad (16b)$$

with  $\nu = \frac{\mu}{12}$ . Eq. (16a) readily indicates that

$$F(\phi)\dot{\phi}^2 = 6\nu^2 a^{-6}(\tau). \quad (17)$$

Also, this equation could be obtained by solving equation (4b). Moreover, the matter contribution of the SB scalar field to the r.h.s. of the Einstein equations would be

$$\rho_\phi = \frac{1}{2}F(\phi)\dot{\phi}^2 \propto a^{-6}. \quad (18)$$

this energy density of a scalar field has the range of scaling behaviors [23, 24], is say, scales exactly as a power of the scale factor like,  $\rho_\phi \propto a^{-m}$ , when the dominant component has an energy density which scales as similar way. So, the contribution of the scalar field is the same as that of stiff matter with a barotropic equation of state  $\gamma = 1$ . This is an interesting result, since the original SB theory was thought of as a way to solve the missing matter problem now generically called the dark matter problem. To solve the latter, one needs a fluid behaving as dust with  $\gamma = 0$ , it is surprising that such a general result remains unnoticed until now in the literature about SB. This is an instance of the results of the analysis of the energy momentum tensor of a scalar field by Marden [25] for General Relativity with scalar matter and by Pimentel [26] for the general scalar tensor theory. In both works a free scalar field is equivalent to a stiff matter fluid.

Furthermore, having identified the general evolution of the scalar field with that of a stiff fluid means that the Eq. (16b) can be integrated separately without a complete solution for the scalar field. In [3] appear a compilation of exact solutions in the case of the original SB theory to FRW cosmological model and in [2] were presented the classical and quantum solution to Bianchi type I.

### III. THE MASTER HAMILTONIAN TO BIANCHI CLASS A COSMOLOGICAL MODELS

Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi Class A cosmological models. The metric has the form

$$ds^2 = -dt^2 + e^{2\Omega(t)} (e^{2\beta(t)})_{ij} \omega^i \omega^j, \quad (19)$$

where  $\beta_{ij}(t)$  is a 3x3 diagonal matrix,  $\beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$ ,  $\Omega(t)$  is a scalar and  $\omega^i$  are one-forms that characterize each cosmological Bianchi type model, and that obey  $d\omega^i = \frac{1}{2}C_{jk}^i \omega^j \wedge \omega^k$ ,  $C_{jk}^i$  the structure constants of the corresponding invariance group, these are included in table 1.

Bianchi type	1-forms $\omega^i$
I	$\omega^1 = dx^1, \quad \omega^2 = dx^2, \quad \omega^3 = dx^3$
II	$\omega^1 = dx^2 - x^1 dx^3, \quad \omega^2 = dx^3, \quad \omega^3 = dx^1$
VI <sub>h=-1</sub>	$\omega^1 = e^{-x^1} dx^2, \quad \omega^2 = e^{x^1} dx^3, \quad \omega^3 = dx^1$
VII <sub>0</sub>	$\omega^1 = dx^2 + dx^3, \quad \omega^2 = -dx^2 + dx^3, \quad \omega^3 = dx^1$
VIII	$\omega^1 = dx^1 + [1 + (x^1)^2]dx^2 + [x^1 - x^2 - (x^1)^2 x^2]dx^3,$ $\omega^2 = 2x^1 dx^2 + (1 - 2x^1 x^2)dx^3,$ $\omega^3 = dx^1 + [-1 + (x^1)^2]dx^2 + [x^1 + x^2 - (x^1)^2 x^2]dx^3$
IX	$\omega^1 = -\sin(x^3)dx^1 + \sin(x^1)\cos(x^3)dx^2,$ $\omega^2 = \cos(x^3)dx^1 + \sin(x^1)\sin(x^3)dx^2, \quad \omega^3 = \cos(x^1)dx^2 + dx^3$

Table 1. *one-forms for the Bianchi Class A models.*

We use the Bianchi type IX cosmological model as toy model to apply method discussed in the previous section. The total Lagrangian density then reads

$$\begin{aligned} \mathcal{L}_{IX} = e^{3\Omega} & \left[ 6 \frac{\dot{\Omega}^2}{N} - 6 \frac{\dot{\beta}_+^2}{N} - 6 \frac{\dot{\beta}_-^2}{N} + \frac{F(\phi)}{N} \dot{\phi}^2 + 16\pi G N \rho - 2N\Lambda \right. \\ & + N e^{-2\Omega} \left\{ \frac{1}{2} \left( e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} \right) \right. \\ & \left. \left. - \left( e^{-2\beta_+ + 2\sqrt{3}\beta_-} + e^{-2\beta_+ - 2\sqrt{3}\beta_-} + e^{4\beta_+} \right) \right\} \right], \quad (20) \end{aligned}$$

making the calculation of momenta in the usual way,  $\Pi_{q^\mu} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}$ , where  $q^\mu = (\Omega, \beta_+, \beta_-, \phi)$

$$\begin{aligned}\Pi_\Omega &= \frac{12}{N}e^{3\Omega}\dot{\Omega}, \quad \rightarrow \quad \dot{\Omega} = \frac{N}{12}e^{-3\Omega}\Pi_\Omega \\ \Pi_+ &= -\frac{12}{N}e^{3\Omega}\dot{\beta}_+, \quad \rightarrow \quad \dot{\beta}_+ = -\frac{N}{12}e^{-3\Omega}\Pi_+ \\ \Pi_- &= -\frac{12}{N}e^{3\Omega}\dot{\beta}_-, \quad \rightarrow \quad \dot{\beta}_- = -\frac{N}{12}e^{-3\Omega}\Pi_- \\ \Pi_\phi &= \frac{2F}{N}e^{3\Omega}\dot{\phi}, \quad \rightarrow \quad \dot{\phi} = \frac{N}{2F}e^{-3\Omega}\Pi_\phi\end{aligned}$$

and introducing into the Lagrangian density, we obtain the canonical Lagrangian as

$$\mathcal{L}_{\text{IX}} = \Pi_{q^\mu}\dot{q}^\mu - N\mathcal{H}_\perp,$$

with the Hamiltonian density

$$\mathcal{H}_\perp = \frac{e^{-3\Omega}}{24} \left( -\Pi_\Omega^2 - \frac{6}{F(\phi)}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 + U(\Omega, \beta_\pm) + C_1 \right), \quad (21)$$

where the gravitational potential becomes,

$$U(\Omega, \beta_\pm) = 12e^{4\Omega} \left( e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{4\beta_+} - 2 \left\{ e^{4\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-} + e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right),$$

with  $C_1 = 384\pi G\rho_1$  corresponding to stiff matter epoch,  $\gamma = 1$ .

The equation (21) can be considered as a master equation for all Bianchi Class A cosmological model in the stiff epoch in the Sáez-Ballester theory, with  $U(\Omega, \beta_\pm)$  is the potential term of the cosmological model under consideration, that can read it to table II.

#### IV. CLASSICAL SCHEME

In this section, we present the classical solutions to all Bianchi Class A cosmological models using the appropriate set of variables,

$$\begin{aligned}\beta_1 &= \Omega + \beta_+ + \sqrt{3}\beta_-, \\ \beta_2 &= \Omega + \beta_+ - \sqrt{3}\beta_-, \\ \beta_3 &= \Omega - 2\beta_+.\end{aligned} \quad (22)$$



Bianchi type	Hamiltonian density $\mathcal{H}$
I	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} \right]$
II	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} + 12e^{4\Omega} e^{4\beta_+ + 4\sqrt{3}\beta_-} \right]$
VI <sub>-1</sub>	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} + 48e^{4\Omega} e^{4\beta_+} \right]$
VII <sub>0</sub>	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} + 12e^{4\Omega} \left( e^{4\beta_+ + 4\sqrt{3}\beta_-} - e^{4\beta_+} + e^{4\beta_+ - 4\sqrt{3}\beta_-} \right) \right]$
VIII	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} + 12e^{4\Omega} \left( e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} - 2 \left\{ e^{4\beta_+} - e^{-2\beta_+ - 2\sqrt{3}\beta_-} - e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right) \right]$
IX	$\frac{e^{-3\Omega}}{24} \left[ -\Pi_\Omega^2 - \frac{6}{F}\Pi_\phi^2 + \Pi_+^2 + \Pi_-^2 - 48\Lambda e^{6\Omega} + 384\pi G\rho_\gamma e^{-3(\gamma-1)\Omega} + 12e^{4\Omega} \left( e^{4\beta_+ + 4\sqrt{3}\beta_-} + e^{4\beta_+ - 4\sqrt{3}\beta_-} + e^{-8\beta_+} - 2 \left\{ e^{4\beta_+} + e^{2\beta_+ - 2\sqrt{3}\beta_-} + e^{-2\beta_+ + 2\sqrt{3}\beta_-} \right\} \right) \right]$

Table II. *Hamiltonian density for the Bianchi Class A models.*

### A. Bianchi I

For building one master equation for all Bianchi Class A models, we begin with the simplest model give by the Bianchi I, and give the general treatment. The corresponding Lagrangian for this cosmological model is written as

$$\mathcal{L}_1 = e^{\beta_1 + \beta_2 + \beta_3} \left[ \frac{2\dot{\beta}_1\dot{\beta}_2}{N} + \frac{2\dot{\beta}_1\dot{\beta}_3}{N} + \frac{2\dot{\beta}_2\dot{\beta}_3}{N} + \frac{F(\phi)\dot{\phi}^2}{N} + 16N\pi G\rho_\gamma e^{-(1+\gamma)(\beta_1 + \beta_2 + \beta_3)} - 2N\Lambda \right], \quad (23)$$

the momenta associated to the variables  $(\beta_i, \phi)$  are

$$\begin{aligned} \Pi_1 &= \frac{2}{N}(\dot{\beta}_2 + \dot{\beta}_3)e^{\beta_1 + \beta_2 + \beta_3}, & \dot{\beta}_1 &= \frac{N}{4}e^{-(\beta_1 + \beta_2 + \beta_3)}(\Pi_2 + \Pi_3 - \Pi_1), \\ \Pi_2 &= \frac{2}{N}(\dot{\beta}_1 + \dot{\beta}_3)e^{\beta_1 + \beta_2 + \beta_3}, & \dot{\beta}_2 &= \frac{N}{4}e^{-(\beta_1 + \beta_2 + \beta_3)}(\Pi_1 + \Pi_3 - \Pi_2), \\ \Pi_3 &= \frac{2}{N}(\dot{\beta}_1 + \dot{\beta}_2)e^{\beta_1 + \beta_2 + \beta_3}, & \dot{\beta}_3 &= \frac{N}{4}e^{-(\beta_1 + \beta_2 + \beta_3)}(\Pi_1 + \Pi_2 - \Pi_3), \\ \Pi_\phi &= \frac{2F\dot{\phi}}{N}e^{\beta_1 + \beta_2 + \beta_3}, & \dot{\phi} &= \frac{N}{2F}e^{-(\beta_1 + \beta_2 + \beta_3)}\Pi_\phi. \end{aligned} \quad (24)$$

so, the Hamiltonian is

$$\begin{aligned} \mathcal{H}_1 &= \frac{1}{8}e^{-(\beta_1 + \beta_2 + \beta_3)} \left[ -\Pi_1^2 - \Pi_2^2 - \Pi_3^2 + \frac{2}{F}\Pi_\phi^2 + 2\Pi_1\Pi_2 + 2\Pi_1\Pi_3 + 2\Pi_2\Pi_3 \right. \\ &\quad \left. + 16\Lambda e^{2(\beta_1 + \beta_2 + \beta_3)} - 128\pi G\rho_\gamma e^{(1-\gamma)(\beta_1 + \beta_2 + \beta_3)} \right], \end{aligned} \quad (25)$$

using the hamilton equation, where  $\iota = \frac{d}{d\tau} = \frac{d}{Nd\tau}$ , we have

$$\Pi'_1 = -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)}, \quad (26)$$

$$\Pi'_2 = -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)}, \quad (27)$$

$$\Pi'_3 = -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)}, \quad (28)$$

$$\Pi'_\phi = \frac{1}{4}e^{-(\beta_1+\beta_2+\beta_3)} \frac{F'}{F^2\phi'} \Pi_\phi^2, \quad (29)$$

$$\beta'_1 = \frac{1}{4}e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_1 + \Pi_2 + \Pi_3], \quad (30)$$

$$\beta'_2 = \frac{1}{4}e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_2 + \Pi_1 + \Pi_3], \quad (31)$$

$$\beta'_3 = \frac{1}{4}e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_3 + \Pi_1 + \Pi_2], \quad (32)$$

$$\phi' = \frac{1}{2F}e^{-(\beta_1+\beta_2+\beta_3)} \Pi_\phi. \quad (33)$$

equations (26,27,28) implies

$$\Pi_1 = \Pi_2 + k_1 = \Pi_3 + k_2. \quad (34)$$

Also, the differential equation for field  $\phi$  can be reduced to quadratures when we use equations (29) and (33), as

$$\frac{1}{2}F(\phi)\phi'^2 = \phi_0 e^{-2(\beta_1+\beta_2+\beta_3)}, \quad \Rightarrow \quad \sqrt{F(\phi)}d\phi = \sqrt{2\phi_0} e^{-(\beta_1+\beta_2+\beta_3)}d\tau, \quad (35)$$

which correspond to equation (5) obtained in direct way from the original Einstein field equation. The corresponding classical solutions for the field  $\phi$  for this cosmological model can be seen in ref. [2].

Using this result and the equation for the field  $\phi$  given in (24) we can find that  $2\frac{\Pi_\phi^2}{F} = 16\phi_0$ . From the hamilton equation for the momenta  $\Pi_1$  can be written for the two equations of state  $\gamma = \pm 1$ , introducing the generic parameter

$$\lambda = \begin{cases} -4\Lambda & \gamma = 1 \\ -4\Lambda + 32\pi G\rho_1 & \gamma = -1 \end{cases} \quad (36)$$

as  $\Pi'_1 = \lambda e^{\beta_1+\beta_2+\beta_3}$ , then re-introducing into the Hamiltonian equation (25) we find one differential equation for the momenta  $\Pi_1$  as

$$\frac{4}{\lambda}\Pi_1'^2 + 2\Pi_1^2 - \kappa\Pi_1 - k_3 = 0, \quad (37)$$

where the corresponding constants are

$$\kappa = 2(k_1 + k_2), \quad k_3 = \begin{cases} k_1^2 + k_2^2 - 16\phi_0, & \gamma = -1 \\ k_1^2 + k_2^2 - 16\phi_0 + 128\pi G\rho_1, & \gamma = 1 \end{cases} \quad (38)$$

and whose solution is

$$\Pi_1 = \frac{\kappa}{6} \pm \frac{\sqrt{\kappa^2 + 12k_3}}{6} \sin \left[ \frac{\sqrt{3\lambda}}{2} \Delta\tau \right]. \quad (39)$$

On the other hand, using this result in the sum of equation (30,31,32), we obtain that

$$\beta_1 + \beta_2 + \beta_3 = \text{Ln} \left[ \frac{\alpha}{\sqrt{12\lambda}} \cos \left[ \frac{\sqrt{3\lambda}}{2} \Delta\tau \right] \right], \quad \alpha = 2\sqrt{\kappa^2 + 12k_3} \quad (40)$$

solution previously found in ref. [2] using the Hamilton-Jacobi approach.

### B. Bianchi's Class A cosmological models

The corresponding Lagrangian for these cosmological model are written using the Lagrangian to Bianchi I, as

$$\mathcal{L}_{\text{II}} = \mathcal{L}_{\text{I}} + N e^{\beta_1 + \beta_2 + \beta_3} \left[ \frac{1}{2} e^{2(\beta_1 - \beta_2 - \beta_3)} \right], \quad (41)$$

$$\mathcal{L}_{\text{VI}_{\text{h}=-1}} = \mathcal{L}_{\text{I}} + N e^{\beta_1 + \beta_2 + \beta_3} \left[ 2e^{-2\beta_3} \right], \quad (42)$$

$$\mathcal{L}_{\text{VII}_{\text{h}=0}} = \mathcal{L}_{\text{I}} + N e^{\beta_1 + \beta_2 + \beta_3} \left[ \frac{1}{2} e^{2(\beta_1 - \beta_2 - \beta_3)} + \frac{1}{2} e^{2(-\beta_1 + \beta_2 - \beta_3)} - e^{-2\beta_3} \right], \quad (43)$$

$$\begin{aligned} \mathcal{L}_{\text{VIII}} = \mathcal{L}_{\text{I}} + \frac{N}{2} e^{\beta_1 + \beta_2 + \beta_3} & \left[ e^{2(\beta_1 - \beta_2 - \beta_3)} + e^{2(-\beta_1 + \beta_2 - \beta_3)} + e^{2(-\beta_1 - \beta_2 + \beta_3)} \right. \\ & \left. - 2 \left( -e^{-2\beta_1} + e^{-2\beta_2} + e^{-2\beta_3} \right) \right], \end{aligned} \quad (44)$$

$$\begin{aligned} \mathcal{L}_{\text{IX}} = \mathcal{L}_{\text{I}} + \frac{N}{2} e^{\beta_1 + \beta_2 + \beta_3} & \left[ e^{2(\beta_1 - \beta_2 - \beta_3)} + e^{2(-\beta_1 + \beta_2 - \beta_3)} + e^{2(-\beta_1 - \beta_2 + \beta_3)} \right. \\ & \left. - 2 \left( e^{-2\beta_1} + e^{-2\beta_2} + e^{-2\beta_3} \right) \right], \end{aligned} \quad (45)$$

the momenta associated to the variables  $(\beta_i, \phi)$  are the same as in equation (64), so, the generic Hamiltonian is

$$\mathcal{H}_{\text{A}} = \mathcal{H}_{\text{I}} - \frac{1}{2} e^{-(\beta_1 + \beta_2 + \beta_3)} [\text{U}_{\text{A}}(\beta_1, \beta_2, \beta_3)], \quad (46)$$

where the potential term  $\text{U}_{\text{A}}(\beta_1, \beta_2, \beta_3)$  is given in table III, where A corresponds to particular Bianchi Class A models (I,II, VI<sub>h=-1</sub>,VII<sub>h=0</sub>,VIII,IX). If we choose the particular gauge to the lapse function  $N = e^{(\beta_1 + \beta_2 + \beta_3)}$ , the equation (46) is much simpler,

$$\mathcal{H}_{\text{A}} = \mathcal{H}_{\text{I}} - \frac{1}{2} [\text{U}_{\text{A}}(\beta_1, \beta_2, \beta_3)], \quad (47)$$

where  $\mathcal{H}_{\text{I}}$  is as in equation (25) but without the factor  $e^{-(\beta_1 + \beta_2 + \beta_3)}$

Bianchi type	Potential $U_A(\beta_1, \beta_2, \beta_3)$
I	0
II	$e^{4\beta_1}$
$VI_{h=-1}$	$4e^{2(\beta_1+\beta_2)}$
$VII_{h=0}$	$e^{4\beta_1} + e^{4\beta_2} - 2e^{2(\beta_1+\beta_2)}$
VIII	$e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} + 2e^{2(\beta_1+\beta_3)} + 2e^{2(\beta_2+\beta_3)}$
IX	$e^{4\beta_1} + e^{4\beta_2} + e^{4\beta_3} - 2e^{2(\beta_1+\beta_2)} - 2e^{2(\beta_1+\beta_3)} - 2e^{2(\beta_2+\beta_3)}$

Table III. *Potential  $U_A(\beta_1, \beta_2, \beta_3)$  for the Bianchi Class A Models.*

The Hamilton equations, for all Bianchi Class A cosmological models are as follows

$$\begin{aligned} \Pi'_1 = & -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)} \\ & + \frac{\partial}{\partial\beta_1} \left( \frac{1}{2} e^{-(\beta_1+\beta_2+\beta_3)} [U_A(\beta_1, \beta_2, \beta_3)] \right) \end{aligned} \quad (48)$$

$$\begin{aligned} \Pi'_2 = & -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)} \\ & + \frac{\partial}{\partial\beta_2} \left( \frac{1}{2} e^{-(\beta_1+\beta_2+\beta_3)} [U_A(\beta_1, \beta_2, \beta_3)] \right), \end{aligned} \quad (49)$$

$$\begin{aligned} \Pi'_3 = & -4\Lambda e^{\beta_1+\beta_2+\beta_3} + 16\pi G(1-\gamma)\rho_\gamma e^{-\gamma(\beta_1+\beta_2+\beta_3)} \\ & + \frac{\partial}{\partial\beta_3} \left( \frac{1}{2} e^{-(\beta_1+\beta_2+\beta_3)} [U_A(\beta_1, \beta_2, \beta_3)] \right), \end{aligned} \quad (50)$$

$$\Pi'_\phi = \frac{1}{4} e^{-(\beta_1+\beta_2+\beta_3)} \frac{F'}{F^2 \phi'} \Pi_\phi^2, \quad (51)$$

$$\beta'_1 = \frac{1}{4} e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_1 + \Pi_2 + \Pi_3], \quad (52)$$

$$\beta'_2 = \frac{1}{4} e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_2 + \Pi_1 + \Pi_3], \quad (53)$$

$$\beta'_3 = \frac{1}{4} e^{-(\beta_1+\beta_2+\beta_3)} [-\Pi_3 + \Pi_1 + \Pi_2], \quad (54)$$

$$\phi' = \frac{1}{2F} e^{-(\beta_1+\beta_2+\beta_3)} \Pi_\phi. \quad (55)$$

In this cosmological models, it is remarkable that the equation for the field  $\phi$  (35) is maintained for all Bianchi Class A models, and in particular, when we use the gauge  $N = e^{\beta_1+\beta_2+\beta_3}$ , the solutions for this field are independent of the cosmological models.

### C. Classical solution in the gauge $N = e^{\beta_1+\beta_2+\beta_3}$ , $\Lambda = 0$ and $\gamma = 1$

With these initial choices, the main equations are written for this gauge as (now a dot means  $\frac{d}{dt}$ )

$$\mathcal{H}_A = \frac{1}{8} \left[ -\Pi_1^2 - \Pi_2^2 - \Pi_3^2 + \frac{2}{F} \Pi_\phi^2 + 2\Pi_1\Pi_2 + 2\Pi_1\Pi_3 + 2\Pi_2\Pi_3 - C_1 \right] - \frac{1}{2} [U_A(\beta_1, \beta_2, \beta_3)], \quad (56)$$

with  $C_1 = 128\pi G\rho_1$ .

The hamilton equation, for all Bianchi Class A cosmological models are

$$\dot{\Pi}_1 = +\frac{\partial}{\partial\beta_1} \left( \frac{1}{2} [U_A(\beta_1, \beta_2, \beta_3)] \right) \quad (57)$$

$$\dot{\Pi}_2 = +\frac{\partial}{\partial\beta_2} \left( \frac{1}{2} [U_A(\beta_1, \beta_2, \beta_3)] \right), \quad (58)$$

$$\dot{\Pi}_3 = +\frac{\partial}{\partial\beta_3} \left( \frac{1}{2} [U_A(\beta_1, \beta_2, \beta_3)] \right), \quad (59)$$

$$\dot{\Pi}_\phi = \frac{1}{4} \frac{\dot{F}}{F^2 \dot{\phi}} \Pi_\phi^2, \quad (60)$$

$$\dot{\beta}_1 = \frac{1}{4} [-\Pi_1 + \Pi_2 + \Pi_3], \quad (61)$$

$$\dot{\beta}_2 = \frac{1}{4} [-\Pi_2 + \Pi_1 + \Pi_3], \quad (62)$$

$$\dot{\beta}_3 = \frac{1}{4} [-\Pi_3 + \Pi_1 + \Pi_2], \quad (63)$$

$$\dot{\phi} = \frac{1}{2F} \Pi_\phi. \quad (64)$$

### 1. Bianchi II

$$\dot{\Pi}_1 = 2e^{4\beta_1} \quad (65)$$

$$\dot{\Pi}_2 = 0, \quad \rightarrow \quad \Pi_2 = p_2 = \text{cte}, \quad (66)$$

$$\dot{\Pi}_3 = 0, \quad \rightarrow \quad \Pi_3 = p_3 = \text{cte}, \quad (67)$$

$$\dot{\Pi}_\phi = \frac{1}{4} \frac{\dot{F}}{F^2 \dot{\phi}} \Pi_\phi^2, \quad (68)$$

$$\dot{\beta}_1 = \frac{1}{4} [-\Pi_1 + p_2 + p_3], \quad (69)$$

$$\dot{\beta}_2 = \frac{1}{4} [-p_2 + \Pi_1 + p_3], \quad (70)$$

$$\dot{\beta}_3 = \frac{1}{4} [-p_3 + \Pi_1 + p_2], \quad (71)$$

$$\dot{\phi} = \frac{1}{2F} \Pi_\phi. \quad (72)$$

introducing (65) into (56) we find the differential equation for  $\Pi_1$  as  $\dot{\Pi}_1 = -\frac{1}{2}\Pi_1^2 + b\Pi_1 + c$  where the constants are defined as  $b = p_2 + p_3$  and  $c = 8\phi_0 - \frac{1}{2}(p_2^2 + p_3^2 + C_1)$ . The solution for  $\Pi_1$  is

$$\Pi_1 = b + \sqrt{-b^2 - 2c} \tan \left[ -\frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right], \quad (73)$$

and the solutions for  $\beta_i$  then are

$$\Delta\beta_1 = -\frac{1}{2} \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right], \quad (74)$$

$$\Delta\beta_2 = \frac{1}{2} p_3 \Delta t + \frac{1}{2} \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right] \quad (75)$$

$$\Delta\beta_3 = \frac{1}{2} p_2 \Delta t + \frac{1}{2} \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right], \quad (76)$$

$$(77)$$

and the solution for the  $\phi$  field is similar to (35)

$$\frac{1}{2} F(\phi) \dot{\phi}^2 = \phi_0, \quad \Rightarrow \quad \sqrt{F(\phi)} d\phi = \sqrt{2\phi_0} dt, \quad (78)$$

So, the solutions in the original variables are

$$\begin{aligned} \Omega &= \frac{1}{6} \left[ (p_2 + p_3) \Delta t + \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right] \right] \\ \beta_- &= \frac{\sqrt{3}}{6} \left[ -\frac{1}{2} p_3 \Delta t - \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right] \right], \\ \beta_+ &= \frac{1}{12} \left[ (p_3 - 2p_2) \Delta t - 2 \text{Log} \left[ \text{Cos} \left( \frac{1}{2} \sqrt{-b^2 - 2c} \Delta t \right) \right] \right]. \end{aligned} \quad (79)$$

## 2. Bianchi VI<sub>h=-1</sub>

$$\dot{\Pi}_1 = 4e^{2(\beta_1 + \beta_2)} \quad (80)$$

$$\dot{\Pi}_2 = 4e^{2(\beta_1 + \beta_2)}, \quad \rightarrow \quad \Pi_2 = \Pi_1 + a_1, \quad (81)$$

$$\dot{\Pi}_3 = 0, \quad \rightarrow \quad \Pi_3 = p_3 = \text{cte}, \quad (82)$$

$$\dot{\Pi}_\phi = \frac{1}{4} \frac{\dot{F}}{F^2 \phi} \Pi_\phi^2, \quad (83)$$

$$\dot{\beta}_1 = \frac{1}{4} [-\Pi_1 + \Pi_2 + p_3], \quad (84)$$

$$\dot{\beta}_2 = \frac{1}{4} [-\Pi_2 + \Pi_1 + p_3], \quad (85)$$

$$\dot{\beta}_3 = \frac{1}{4} [-p_3 + \Pi_1 + \Pi_2], \quad (86)$$

$$\dot{\phi} = \frac{1}{2F} \Pi_\phi. \quad (87)$$

introducing (81) into (56) we find the differential equation for  $\Pi_1$  as  $\dot{\Pi}_1 - p_3 \Pi_1 + k_1 = 0$  where  $k_1 = \frac{1}{4} (p_3^2 + a_1^2 - 16\phi_0 + C_1 - 2a_1 p_3)$  who solution become as

$$\Pi_1 = \frac{1}{p_3} [e^{p_3 \Delta t} + k_1], \quad (88)$$

then the solutions for  $\beta_i$  become

$$\Delta \beta_1 = \frac{1}{4} (a_1 + p_3) \Delta t, \quad (89)$$

$$\Delta \beta_2 = \frac{1}{4} (p_3 - a_1) \Delta t, \quad (90)$$

$$\Delta \beta_3 = \frac{1}{4} (a_1 - p_3) \Delta t + \frac{1}{2p_3} [e^{p_3 \Delta t} + k_1], \quad (91)$$

$$(92)$$

and the solutions in the original variables are

$$\begin{aligned} \Omega &= \frac{1}{12p_3} [2k_1 + p_3 (a_1 + p_3) \Delta t + 2e^{p_3 \Delta t}], \\ \beta_1 &= \frac{a_1}{4\sqrt{3}} \Delta t, \\ \beta_+ &= -\frac{1}{12p_3} [2k_1 + p_3 (a_1 - 2p_3) \Delta t + 2e^{p_3 \Delta t}]. \end{aligned} \quad (93)$$

## V. QUANTUM SCHEME

The WDW equation for these models is achieved by replacing  $\Pi_{q^\mu} = -i\partial_{q^\mu}$  in (21). The factor  $e^{-3\Omega}$  may be factor ordered with  $\hat{\Pi}_\Omega$  in many ways. Hartle and Hawking [27] have suggested what might be called a semi-general factor ordering which in this case would order  $e^{-3\Omega} \hat{\Pi}_\Omega^2$  as

$$\begin{aligned} -e^{-(3-Q)\Omega} \partial_\Omega e^{-Q\Omega} \partial_\Omega &= -e^{-3\Omega} \partial_\Omega^2 + Q e^{-3\Omega} \partial_\Omega, \\ -\frac{6}{F} \phi^s \frac{\partial}{\partial \phi} \phi^{-s} \frac{\partial}{\partial \phi} &= -\frac{6}{F} \frac{\partial^2}{\partial \phi^2} + \frac{6s}{F} \phi^{-1} \frac{\partial}{\partial \phi} \end{aligned} \quad (94)$$

where  $Q$  and  $s$  are any real constants that measure the ambiguity in the factor ordering in the variables  $\Omega$  and  $\phi$ . We will assume in the following this factor ordering for the Wheeler-DeWitt equation, which becomes

$$\square \Psi - \frac{6}{F(\phi)} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{6s}{F} \phi^{-1} \frac{\partial \Psi}{\partial \phi} + Q \frac{\partial \Psi}{\partial \Omega} - U(\Omega, \beta_\pm) \Psi - C_1 \Psi = 0, \quad (95)$$

where  $\square$  is the three dimensional d'Lambertian in the  $\ell^\mu = (\Omega, \beta_+, \beta_-)$  coordinates, with signature  $(- + +)$ .

When we introduce the Ansatz  $\Psi = \chi(\phi)\psi(\Omega, \beta_{\pm})$  in (95), we obtain the general set of differential equations (under the assumed factor ordering) for the Bianchi type IX cosmological model

$$\square \psi + Q \frac{\partial \psi}{\partial \Omega} - [U(\Omega, \beta_{\pm}) + C_1 - \mu^2] \psi = 0, \quad (96)$$

$$\frac{6}{F(\phi)} \frac{\partial^2 \chi}{\partial \phi^2} - \frac{6s}{F} \phi^{-1} \frac{\partial \chi}{\partial \phi} + \mu^2 \chi = 0 \quad (97)$$

When we calculate the solution to equation (97), we find interesting properties on this, as

1. This equation is a master equation for the field  $\phi$  for any cosmological model, implying that this field  $\phi$  is an universal field as cosmic ground, having the best presence in the stiff matter era as an ingredient in the formation the structure galaxies and when we consider two types of functions,  $F(\phi) = \omega \phi^m$  and  $F(\phi) = \omega e^{m\phi}$ , we have the following exact solutions [28]

(a)  $F(\phi) = \omega \phi^m$

the differential equation to solver is

$$\frac{d^2 \chi}{d\phi^2} - s \phi^{-1} \frac{d\chi}{d\phi} + \alpha \phi^m \chi = 0 \quad (98)$$

with  $\alpha = \frac{\omega \mu^2}{6}$ . The solutions depend on the value to  $m$  and  $s$ ,

- i. General solution for any  $m \neq -2$  and  $s$ , are written in terms of ordinary and modify Bessel function,

$$\chi = c_1 \phi^{\frac{1+s}{2}} Z_{\nu} \left( \frac{2\sqrt{\alpha}}{m+2} \phi^{\frac{m+2}{2}} \right), \quad (99)$$

with  $c_1$  an integration constant,  $Z_{\nu}$  is a generic Bessel function,  $\nu = \frac{1+s}{m+2}$  is the order. When  $\alpha > 0$  imply  $\omega > 0$ ,  $Z_{\nu}$  become the ordinary Bessel function,  $(J_{\nu}, Y_{\nu})$ .

If  $\alpha < 0$ ,  $\rightarrow w < 0$ ,  $Z_{\nu} \rightarrow (I_{\nu}, K_{\nu})$ .

- ii.  $m = -2$  and any  $s$ ,

$$\chi = \phi^{\frac{1+s}{2}} \begin{cases} c_1 \phi^{\mu} + c_2 \phi^{-\mu} & \text{si } \mu > 0 \\ c_1 + c_2 \ln \phi & \text{si } \mu = 0 \\ c_1 \sin(\mu \ln \phi) + c_2 \cos(\mu \ln \phi) & \text{if } \mu < 0 \end{cases} \quad (100)$$

where  $\mu = \frac{1}{2} \sqrt{|(1+s)^2 - 4\alpha|}$

- iii.  $m = -6$  and  $s = 1$

$$\chi(\phi) = \phi^2 \begin{cases} c_1 \sinh \left( \frac{\sqrt{|\alpha|}}{2\phi^2} \right) + c_2 \cosh \left( \frac{\sqrt{|\alpha|}}{2\phi^2} \right) & \alpha < 0 \rightarrow \omega < 0 \\ c_1 \sin \left( \frac{\sqrt{|\alpha|}}{2\phi^2} \right) + c_2 \cos \left( \frac{\sqrt{|\alpha|}}{2\phi^2} \right) & \alpha > 0 \rightarrow \omega > 0 \end{cases} \quad (101)$$



(b)  $F(\phi) = \omega e^{m\phi}$ , for this case we consider the caso  $s = 0$ ,

$$\frac{d^2\chi}{d\phi^2} + \alpha e^{m\phi}\chi = 0 \quad (102)$$

i.  $m \neq 0$

$$\chi = CZ_0\left(\frac{2\sqrt{\alpha}}{m}e^{\frac{m\phi}{2}}\right) \quad (103)$$

with  $C$  is a integration constant and  $Z_0$  is the generic Bessel function to zero order. So, if  $\alpha > 0$  then  $\omega > 0$ ,  $Z_0$  is the ordinary Bessel function  $(J_0, Y_0)$ . When  $\alpha < 0, \rightarrow \omega < 0$ ,  $Z_0 \rightarrow (I_0, K_0)$ .

ii. for  $m = 0$ ,

$$\chi = \begin{cases} c_1 \sinh(\sqrt{|\alpha|}\phi) + c_2 \cosh(\sqrt{|\alpha|}\phi) & \text{if } \alpha < 0 \rightarrow \omega < 0 \\ c_1 \sin(\sqrt{|\alpha|}\phi) + c_2 \cos(\sqrt{|\alpha|}\phi) & \text{if } \alpha > 0 \rightarrow \omega > 0 \end{cases} \quad (104)$$

2. If we have the solution for the parameter  $s=0$  for arbitrary function  $F(\phi)$ , say  $\chi_0$ , then we have also the solution for  $s=-2$ , as  $\chi(s=-2) = \frac{\chi_0}{\phi}$ .

To obtain the solution of the other factor of  $\Psi$  we use the particular value for the constants  $C_1 = \mu^2$ , and make the following Ansatz for the wave function

$$\psi(\ell^\mu) = W(\ell^\mu)e^{-S(\ell^\mu)}, \quad (105)$$

where  $S(\ell^\mu)$  is known as the superpotential function, and  $W$  is the amplitude of probability to that employed in Bohmian formalism [29], those found in the literature, years ago [22]. So (96) is transformed into

$$\square W - W\square S - 2\nabla W \cdot \nabla S + Q\frac{\partial W}{\partial\Omega} - QW\frac{\partial S}{\partial\Omega} + W[(\nabla S)^2 - U] = 0, \quad (106)$$

where  $\square = G^{\mu\nu} \frac{\partial^2}{\partial\ell^\mu \partial\ell^\nu}$ ,  $\nabla W \cdot \nabla \Phi = G^{\mu\nu} \frac{\partial W}{\partial\ell^\mu} \frac{\partial \Phi}{\partial\ell^\nu}$ ,  $(\nabla)^2 = G^{\mu\nu} \frac{\partial}{\partial\ell^\mu} \frac{\partial}{\partial\ell^\nu} = -(\frac{\partial}{\partial\Omega})^2 + (\frac{\partial}{\partial\beta_+})^2 + (\frac{\partial}{\partial\beta_-})^2$ , with  $G^{\mu\nu} = \text{diag}(-1, 1, 1)$ ,  $U$  is the potential term of the cosmological model under consideration.

Eq (106) can be written as the following set of partial differential equations

$$(\nabla S)^2 - U = 0, \quad (107a)$$

$$W\left(\square S + Q\frac{\partial S}{\partial\Omega}\right) + 2\nabla W \cdot \nabla S = 0, \quad (107b)$$

$$\square W + Q\frac{\partial W}{\partial\Omega} = 0. \quad (107c)$$

Following reference [30], first we shall choose to solve Eqs. (107a) and (107b), whose solutions at the end will have to fulfill Eq. (107c), which play the role of a constraint equation.

### A. Transformation of the Wheeler-DeWitt equation

We were able to solve (107a), by doing the change of coordinates (22) and rewrite (107a) in these new coordinates. With this change, the function  $S$  is obtained as follow, with the ansatz (105),

In this section, we obtain the solutions to the equations that appear in the decomposition of the WDW equation, (107a), (107b) and (107c), using the Bianchi type IX Cosmological model. So, the equation  $[\nabla]^2 = -(\frac{\partial}{\partial\Omega})^2 + (\frac{\partial}{\partial\beta_+})^2 + (\frac{\partial}{\partial\beta_-})^2$  can be written in the following way (see appendix section 9)

$$\begin{aligned} [\nabla]^2 &= 3 \left[ \left( \frac{\partial}{\partial\beta_1} \right)^2 + \left( \frac{\partial}{\partial\beta_2} \right)^2 + \left( \frac{\partial}{\partial\beta_3} \right)^2 \right] - 6 \left[ \frac{\partial}{\partial\beta_1} \frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_1} \frac{\partial}{\partial\beta_3} + \frac{\partial}{\partial\beta_2} \frac{\partial}{\partial\beta_3} \right] \\ &= 3 \left( \frac{\partial}{\partial\beta_1} + \frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_3} \right)^2 - 12 \left[ \frac{\partial}{\partial\beta_1} \frac{\partial}{\partial\beta_2} + \frac{\partial}{\partial\beta_1} \frac{\partial}{\partial\beta_3} + \frac{\partial}{\partial\beta_2} \frac{\partial}{\partial\beta_3} \right]. \end{aligned} \quad (108)$$

The potencial term of the Bianchi type IX is transformed in the new variables into

$$U = 12 \left[ \left( e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} \right)^2 - 2e^{2(\beta_1+\beta_2)} - 2e^{2(\beta_1+\beta_3)} - 2e^{2(\beta_2+\beta_3)} \right] \quad (109)$$

Then (107a) for this models is rewritten in the new variables as

$$\begin{aligned} &3 \left( \frac{\partial S}{\partial\beta_1} + \frac{\partial S}{\partial\beta_2} + \frac{\partial S}{\partial\beta_3} \right)^2 - 12 \left[ \frac{\partial S}{\partial\beta_1} \frac{\partial S}{\partial\beta_2} + \frac{\partial S}{\partial\beta_1} \frac{\partial S}{\partial\beta_3} + \frac{\partial S}{\partial\beta_2} \frac{\partial S}{\partial\beta_3} \right] \\ &- 12 \left[ \left( e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} \right)^2 - 4e^{2(\beta_1+\beta_2)} - 4e^{2(\beta_1+\beta_3)} - 4e^{2(\beta_2+\beta_3)} \right] = 0. \end{aligned} \quad (110)$$

Now, we can use the separation of variables method to get solutions to the last equation for the  $S$  function, obtaining for the Bianchi type IX model

$$S_{IX} = \pm \left( e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} \right). \quad (111)$$

In table IV we present the corresponding superpotential function  $S$  and amplitude  $W$  for all Bianchi Class A models.

With this result, and using for the solution to (107b) in the new coordinates  $\beta_i$ , we have for  $W$  function as

$$W_{IX} = W_0 e^{[(1+\frac{Q}{6})(\beta_1+\beta_2+\beta_3)]}. \quad (112)$$

and re-introducing this result into Eq. (107c) we find that  $Q = \pm 6$ . Therefore we have two wave

functions

$$\begin{aligned}
\psi_{\text{IX}}(\beta_i) &= W_{\text{IX}}(\beta_i) \text{Exp} \left[ \pm \left( e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} \right) \right] \\
&= \text{Exp} \left[ \pm \left( e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3} \right) \right] \begin{cases} W_0, & Q=-6, \\ W_0 \text{Exp} [2(\beta_1 + \beta_2 + \beta_3)], & Q=6 \end{cases} \quad (113)
\end{aligned}$$

similar solutions were given by Moncrief and Ryan [31] in standard quantum cosmology in general relativity. In table IV we present the superpotential function S, the amplitude of probability W and the relations between the parameters for the corresponding Bianchi Class A models.

Bianchi type	Superpotential S	Amplitude of probability W	Constraint
I	constant	$e^{(\frac{r}{3} + \frac{b}{6} + \frac{\sqrt{3}c}{6})\beta_1 + (\frac{r}{3} + \frac{b}{6} - \frac{\sqrt{3}c}{6})\beta_2 + (\frac{r}{3} - \frac{b}{3})\beta_3}$	$r^2 - Qr - a^2 = 0,$ $a^2 = b^2 + c^2$
II	$e^{2\beta_1}$	$e^{(a-1-\frac{Q}{6})\beta_1 + a\beta_2 + (a-b)\beta_3}$	$144b^2 - 144ab + 36$ $-Q^2 + 24aQ = 0$
VI <sub>h=-1</sub>	$2(\beta_1 - \beta_2) e^{(\beta_1 + \beta_2)}$	$e^{a(\beta_1 + \beta_2)}$	$Q = 0$
VII <sub>h=0</sub>	$e^{2\beta_1} + e^{2\beta_2}$	$e^{(1+\frac{Q}{6})(\beta_1 + \beta_2 + \beta_3) + a(\beta_1 + \beta_2)}$	$Q^2 - 48a - 36 = 0$
VIII	$e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3}$	$W_0 e^{[(1+\frac{Q}{6})(\beta_1 + \beta_2 + \beta_3)]}$	$Q = \pm 6$
IX	$e^{2\beta_1} + e^{2\beta_2} + e^{2\beta_3}$	$W_0 e^{[(1+\frac{Q}{6})(\beta_1 + \beta_2 + \beta_3)]}$	$Q = \pm 6$

Table IV. *superpotential S, the amplitude of probability W and the relations between the parameters for the corresponding Bianchi Class A models.*

If one looks at the expressions for the functions S given in table IV, one notes that there is a general form to write them using the 3x3 matrix  $m^{ij}$  that appear in the classification scheme of Ellis and MacCallum [32] and Ryan and Shepley [33], the structure constants are written in the form

$$C_{jk}^i = \epsilon_{jks} m^{si} + \delta_{[k}^i a_{j]}. \quad (114)$$

where  $a_i = 0$  for the Class A models.

If we define  $g_i(\beta_i) = (e^{\beta_1}, e^{\beta_2}, e^{\beta_3})$ , with  $\beta_i$  given in (22), the solution to (107a) can be written as

$$S(\beta_i) = \pm [g_i M^{ij} (g_j)^T]. \quad (115)$$

where  $M^{ij} = m^{ij}$  for the Bianchi Class A, excepting the Bianchi type  $VI_{h=-1}$  for which we redefine the matrix to be consistent with (115)

$$M^{ij} = (\beta_1 - \beta_2) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, for the Bianchi Class A models, the wave function  $\Psi$  can be written in the general form

$$\Psi = \chi(\phi) W(\beta_i) \exp [\pm [g_i M^{ij} (g_j)^T]]. \quad (116)$$

## VI. FINAL REMARKS

Using the analytical procedure of hamilton equation of classical mechanics, in appropriate coordinates, we found a master equation for all Bianchi Class A cosmological models, we present partial result in the classical regime for three models of them, but the general equation are shown for all them. In particular, the Bianchi type I is complete solved without using a particular gauge. The Bianchi type II and  $VI_{h=-1}$  are solved introducing a particular gauge. An important results yields when we use the gauge  $N = e^{\beta_1 + \beta_2 + \beta_3}$ , we find that the solutions for the  $\phi$  field are independent of the cosmological models, and we find that the energy density associated has a scaling behaviors under the analysis of standard field theory to scalar fields [23, 24], is say, scales exactly as a power of the scale factor like,  $\rho_\phi \propto a^{-m}$ . More of this can be seen to references cited before. On the other hand, in the quantum regime, wave functions of the form  $\Psi = W e^{\pm S}$  are the only known exact solutions for the Bianchi type IX model in standard quantum cosmology. In the SB formalism, these solutions are modified only for the function  $\chi$ ,  $\Psi = \chi(\phi) W(\ell^\mu) e^{\pm S(\ell^\mu)}$  when we include the particular ansatz  $C_1 = \mu^2$ . This kind of solutions already have been found in supersymmetric quantum cosmology [34] and also for the WDW equation defined in the bosonic sector of the heterotic strings [35]. Recently, in the books [36] appears all solutions in the supersymmetric scheme similar at our formalism. We have shown that they are also exact solutions to the rest of the Bianchi Class A models in SB quantum cosmology, under the assumed semi-general factor ordering (94). Different procedures seem to produce this particular quantum state, where  $S$  is a solution to the corresponding classical Hamilton-Jacobi equation (107a).

## VII. APPENDIX: ENERGY MOMENTUM TENSOR

From Eq. (6) we see that the effective energy momentum tensor of the scalar field is

$$T_{\alpha\beta} = F(\phi) \left( \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma} \phi^{,\gamma} \right) \quad (117)$$

this energy momentum tensor is conserved, as follows from the equation of motion for the scalar field

$$\begin{aligned} \nabla^\beta T_{\alpha\beta} &= \nabla^\beta \left[ F(\phi) \left( \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma} \phi^{,\gamma} \right) \right] = F'(\phi) \phi^{,\beta} \left( \phi_{,\alpha} \phi_{,\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma} \phi^{,\gamma} \right) \\ &\quad + F(\phi) \left( \phi_{,\alpha}^{;\beta} \phi_{,\beta} + \phi_{,\alpha} \phi_{,\beta}^{;\beta} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma}^{;\beta} \phi^{,\gamma} - \frac{1}{2} g_{\alpha\beta} \phi_{,\gamma} \phi^{,\gamma;\beta} \right) \\ &= F'(\phi) \left( \frac{1}{2} \phi_{,\gamma} \phi^{,\gamma} \phi_{,\alpha} \right) + F(\phi) \left( \phi_{,\alpha}^{;\beta} \phi_{,\beta} + \phi_{,\alpha} \phi_{,\beta}^{;\beta} - g_{\alpha\beta} \phi_{,\gamma}^{;\beta} \phi^{,\gamma} \right) \\ &= \frac{1}{2} \phi_{,\alpha} \left( F'(\phi) \phi_{,\gamma} \phi^{,\gamma} + 2F(\phi) \phi_{,\alpha} \phi_{,\beta}^{;\beta} \right) = 0 \end{aligned} \quad (118)$$

Now we proceed to show that the energy momentum tensor has the structure of an imperfect stiff fluid,

$$T_{\alpha\beta} = (\rho + p) U_\alpha U_\beta + p g_{\alpha\beta} = (2\rho) [U_\alpha U_\beta + \frac{1}{2} g_{\alpha\beta}] \quad (119)$$

here  $\rho$  is the energy density,  $p$  the pressure, and  $U_\alpha$  the velocity. If we choose for the velocity the normalized derivative of the scalar field, assuming that it is a timelike vector, as is often the case in cosmology, where the scalar field is only time dependent

$$U_\alpha = S^{-1/2} \phi_{,\alpha}, \quad S = -\phi_{,\sigma} \phi^{,\sigma}, \quad (120)$$

It is evident that the energy momentum tensor of the SB theory is equivalent to a stiff fluid with the energy density given by

$$\rho = \frac{S F(\phi)}{2} = -\frac{\phi_{,\sigma} \phi^{,\sigma} F(\phi)}{2}. \quad (121)$$

Therefore the most important contribution of the scalar field occurs during a stiff matter phase that is previous to the dust phase.

## VIII. APPENDIX: OPERATORS IN THE $\beta_i$ VARIABLES

The operators who appear in eqn (95) are calculated in the original variables  $(\Omega, \beta_+, \beta_-)$ ; however the structure of the cosmological potential term gives us an idea to implement new variables,

considering the Bianchi type IX cosmological model, these one given by eqn (22). The main calculations are based in the following

$$\begin{aligned}
\frac{\partial}{\partial \Omega} &= \frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_3}, \\
\frac{\partial^2}{\partial \Omega^2} &= \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} + \frac{\partial^2}{\partial \beta_3^2} + 2 \left[ \frac{\partial^2}{\partial \beta_1 \partial \beta_2} + \frac{\partial^2}{\partial \beta_1 \partial \beta_3} + \frac{\partial^2}{\partial \beta_2 \partial \beta_3} \right], \\
\frac{\partial}{\partial \beta_+} &= \frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} - 2 \frac{\partial}{\partial \beta_3}, \\
\frac{\partial^2}{\partial \beta_+^2} &= \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} + 4 \frac{\partial^2}{\partial \beta_3^2} + 2 \left[ \frac{\partial^2}{\partial \beta_1 \partial \beta_2} - 2 \frac{\partial^2}{\partial \beta_1 \partial \beta_3} - 2 \frac{\partial^2}{\partial \beta_2 \partial \beta_3} \right], \\
\frac{\partial}{\partial \beta_-} &= \sqrt{3} \left( \frac{\partial}{\partial \beta_1} - \frac{\partial}{\partial \beta_2} \right), \\
\frac{\partial^2}{\partial \beta_-^2} &= 3 \left( \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} - 2 \frac{\partial^2}{\partial \beta_1 \partial \beta_2} \right).
\end{aligned} \tag{122}$$

So, the operator  $(\nabla)^2$ ,  $\square$ ,  $\nabla S \nabla W$  are written as

$$\begin{aligned}
(\nabla)^2 &= G^{\mu\nu} \frac{\partial}{\partial \ell^\mu} \frac{\partial}{\partial \ell^\nu}, \quad G^{\mu\nu} = \text{diag}(-1, 1, 1), \quad \ell^\mu = (\Omega, \beta_+, \beta_1), \\
&= 3 \left\{ \left( \frac{\partial}{\partial \beta_1} \right)^2 + \left( \frac{\partial}{\partial \beta_2} \right)^2 + \left( \frac{\partial}{\partial \beta_3} \right)^2 - 2 \left[ \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_3} + \frac{\partial}{\partial \beta_2} \frac{\partial}{\partial \beta_3} \right] \right\} \\
&= 3 \left\{ \left[ \frac{\partial}{\partial \beta_1} + \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_3} \right]^2 - 4 \left[ \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \beta_1} \frac{\partial}{\partial \beta_3} + \frac{\partial}{\partial \beta_2} \frac{\partial}{\partial \beta_3} \right] \right\} \\
\square &= G^{\mu\nu} \frac{\partial^2}{\partial \ell^\mu \partial \ell^\nu} = 3 \left( \frac{\partial^2}{\partial \beta_1^2} + \frac{\partial^2}{\partial \beta_2^2} + \frac{\partial^2}{\partial \beta_3^2} \right) - 6 \left( \frac{\partial^2}{\partial \beta_1 \partial \beta_2} + \frac{\partial^2}{\partial \beta_1 \partial \beta_3} + \frac{\partial^2}{\partial \beta_2 \partial \beta_3} \right) \\
\nabla S \cdot \nabla W &= G^{\mu\nu} \frac{\partial S}{\partial \ell^\mu} \frac{\partial W}{\partial \ell^\nu}, \\
&= 3 \left( \frac{\partial S}{\partial \beta_1} \frac{\partial W}{\partial \beta_1} + \frac{\partial S}{\partial \beta_2} \frac{\partial W}{\partial \beta_2} + \frac{\partial S}{\partial \beta_3} \frac{\partial W}{\partial \beta_3} \right) \\
&\quad - 3 \left( \frac{\partial S}{\partial \beta_1} \frac{\partial W}{\partial \beta_2} + \frac{\partial S}{\partial \beta_1} \frac{\partial W}{\partial \beta_3} + \frac{\partial S}{\partial \beta_2} \frac{\partial W}{\partial \beta_3} + \frac{\partial S}{\partial \beta_2} \frac{\partial W}{\partial \beta_1} + \frac{\partial S}{\partial \beta_3} \frac{\partial W}{\partial \beta_1} + \frac{\partial S}{\partial \beta_3} \frac{\partial W}{\partial \beta_2} \right)
\end{aligned} \tag{123}$$

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